UNIVERSITY OF RIJEKA

FACULTY OF ENGINEERING

CONSTITUTIVE MODELING AND MATERIAL BEHAVIOR

Interim Report

~work in progress~

RIJEKA, 2016.

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Report within the framework of:	Scientific Project Number IP-2014-09-4982, Croatian Science Foundation *Development of evolutionary procedures for characterization of biological tissues behavior BIOMAT*
Authors:	Assoc. Prof. Robert Basan, D.Sc. Tea Marohnić, M.Sc.
Abstract:	This report presents systematic overview of fundamentals of constitutive modeling and material behavior. Classification of real material behavior is given, and described in brief. Further on, plastic behavior of materials is described in more detail, along with fundamentals of elastoplastic material behavior modeling. Information on complex material models available in ANSYS software is given. So far, nonlinear materials models dealing with rate-independent metal plasticity are covered, with the aim of expanding the content to materials other than metals during the second year of the project.

RIJEKA, 2016.

CONTENTS

1. INTRODUCTION

In today’s world of global competition, the goal of successful product development is reducing the time and money needed to design, manufacture and place the product on global market. Considering the influence on costs is largest in early design phases, calculations of carrying capacity and durability of components and structures, and computer simulations of product behavior during operation are made in order to prevent (or at least seriously reduce the possibility of) expensive and time-consuming changes in later phases of product development.

Evaluation and selection of material, together with the knowledge of its behavior, is one of the most important decisions to be made in early design phases – not only because the material costs have a significant share in total costs of product development, but also because the previously mentioned calculations and simulations cannot be performed without it.

Up to 80% of failures of engineering components are caused by dynamic load. Experimental characterization of material behavior, when loaded dynamically, is, of course, the most accurate, but also expensive, complicated and time-consuming and often not available in early design phases. For this reason we strive to mathematically represent and predict the complex material behavior under dynamic loading. For most engineering applications, such material behavior can be quite accurately represented with relatively simple mathematical formulations based on the stabilized behavior of material. These require a modest number of material parameters to fit the experimental data. Such parameters can be estimated well enough on the basis of monotonic parameters of materials.

However, if we want to characterize the material behavior from the beginning of loading up to finale failure, or if the material behaves in non-stabilized manner, more complex mathematical formulations are needed. The process of obtaining complex models of material behavior is called the **constitutive modeling.**

2. CONSTITUTIVE MODELING AND MATERIAL BEHAVIOR

2.1. CONCEPT AND ROLE OF CONSTITUTIVE MODELING

Every operating condition, especially the severe one such as higher mechanical loadings or increased temperature, influences the engineering material in use and causes deterioration of material properties, due to, usually concurrent, processes of deformation and damage [1]. Material properties are thus not the same as of the "virgin" material. This can cause a failure of an engineering component, or even worse – the whole structure. In order to prevent unexpected events, which may have very serious consequences such as human victims or nuclear catastrophe, calculations and engineering analysis must be performed for prediction of a safe lifetime of components and structures. Most of the practical methods for predicting lifetime are based on empiricism – thus, large amount of experimental data is needed for reliable predictions. Durability calculations are performed in early design phases when experimental data is rarely available (due to demanding, expensive and long lasting experiments). Hence, it is quite useful to be able to predict material response to the applied loading (deformation behavior), or any other occurrence of interest. This is the task of constitutive modeling.

Constitutive models differ for various types of engineering materials (metals and alloys, polymers, concrete, wood etc.) since physical mechanisms that cause deterioration of material at macroscopic (observed) level are entirely different. Constitutive models are the mathematical simplification of this complex physical behavior. Despite the fact that material properties are determined by material microstructure, they all show, to a greater or lesser extent, the similar mechanical behavior (elasticity, yielding, plastic strain, strain induced anisotropy, hysteresis loops when cyclically loaded, damage by monotonic loading or fatigue, and crack growth under static or dynamic loads). Therefore, it is possible to successfully develop models of the common behavior of materials with the means of continuum mechanics and the thermodynamics of irreversible processes, without detailed reference to the complexity of their physical microstructures.

It must also be emphasized that **there is no such thing as an ‘‘exact’’ model for a particular material** since, except for the linear phenomena, **there is no unique way to build a constitutive model for a certain behavior**. Consider the behavior of steel. It can be represented by an elastoplastic model, but it is quite inappropriate to attribute steel as elastoplastic.

For example, steel can be modeled in following ways [2]:

- structural analysis under working load: linear elasticity
- analysis of damped vibrations: viscoelasticity
- calculation of limit load: rigid perfect plasticity
- accurate calculation of permanent deformation after monotonic and cyclic loading: hardening elastoplasticity
- analysis of stationary creep and relaxation: perfect (nonhardening) elasto-viscoplasticity
- prediction of lifetime in high-cycle-fatigue: damage coupled to elastic deformations
- prediction of lifetime in low-cycle-fatigue: damage coupled to plastic deformations
- prediction of lifetime in creep and creep-fatigue: damage coupled to viscoplastic deformations
- prediction of stability of a preexisting crack: linear elasticity (from which singular stress fields are derived for sharp cracks)
- prediction of strain localization in shear bands and incipient material failure: softening plasticity or damage coupled to plastic deformation.

Since the mathematical structure of a model depends not only on the material, but also on its purpose (i.e. operating conditions) and required accuracy, constitutive modeling can be, in a way, considered as an art. Choosing the right model depends strongly on the skills and experience of the scientist whose task is to choose a model that:

- is relevant for describing the physical phenomena at hand,
- provides sufficiently accurate prediction for given application,
- can be implemented in a robust mechanical algorithm for obtaining the truly operational model.

2.2. APPROACHES TO CONSTITUTIVE MODELING

2.2.1. Fundamental (or micromechanics) approach

Fundamental approach to constitutive modeling is based on a thorough understanding of microstructural processes that cause deformation and failure in a material. Elementary constitutive relations are established for the microstructural behavior, hence the name **micromechanical modeling** [2]. The macroscopic behavior is the result of averaging techniques (homogenization) which can sometimes be carried out analytically, but more generally is carried out numerically with the aid of Representative Volume Element (RVE).

By RVE we mean a volume of a size large enough to represent material heterogeneities, yet small enough to represent a ‘‘point’’ so that application of continuum mechanics equations is justified. The size of RVE is often determined by periodicity of the microstructural arrangement, which is about $(0,1\text{mm})^3$ for metallic materials. Besides the materials with ordered lattice structures, also the ‘‘disordered’’ media (soil, rock, etc.) is considered by this approach.

The direct identification of materials by the fundamental approach is complex due to difficulties in measuring microscopic variables needed to develop a model (density of dislocation, density of cavities, texture etc.), so that fundamental approach to constitutive modeling is still under development.

2.2.2. Phenomenological approach

The mostly used approach – the **phenomenological** one, is based on studying the behavior of the volume element of matter subjected to elementary tests. Macroscopic models are established directly from these observations, without taking into account the complex microstructural processes. Basically, mathematical function that represents the nonlinear relationship between stress and strain is established on the basis of **observable**, i.e. physically available variables (such as stress, strain, temperature and time) and additional, **nonobservable (internal)** variables that represent the overall influence of the microstructural changes to the volume element. Internal variables are thus defined at the macroscale level.

The goal of model calibration is, of course, minimization of the difference between predicted response and the experimentally obtained data.

2.2.3. Statistical approach

The third and the least fundamental approach for describing material behavior is the **statistical approach**. Statistical models are normally established as response functions for specific combination of loading and environmental conditions, and their applicability beyond those particular conditions is highly questionable.

2.3. MATERIAL BEHAVIOR MODELS

Selecting the right equations to describe material behavior is the crucial part of setting up a solid mechanics analysis [3]. Wrong model used, or the inaccurate material properties will lead to a completely invalid predictions. It has been said already that the constitutive model for a material is a set of equations relating stress to strain (and possibly strain history, strain rate, and other field quantities) and fit to experimental measurements (phenomenological approach).

Regardless of the type of material and the physical mechanisms that act when the material is subjected to certain input, the response of materials to characteristic tests allows us to classify them by following adjectives: rigid, elastic, viscous, plastic and perfectly plastic [4].

Classification of real behavior of materials is presented in Figure 1.

Type of solid	Material behavior	Rate dependency
elastic	perfectly elastic solid	rate-independent
	viscoelastic solid	rate-dependent
plastic	rigid-perfectly plastic solid	rate-independent
	elastic perfectly plastic solid	
	elastoplastic hardening solid	
viscoplastic	perfectly viscoplastic solid	rate-dependent
	elastic perfectly viscoplastic solid	
	elastic viscoplastic hardening solid	

Figure 1 Material behavior classification

2.3.1. Elastic solids

Perfectly elastic solid

If you conduct a uniaxial tensile test on almost any material, and keep the stress level sufficiently low (below the elastic limit), you will observe that the specimen deforms reversibly: if the loads are removed, the solid returns to its original shape. The strain in the specimen depends only on the stress applied to it and it doesn’t depend on the rate of loading, or the history of loading. Elastic response can be linear as well as nonlinear, as shown in Figure 2.

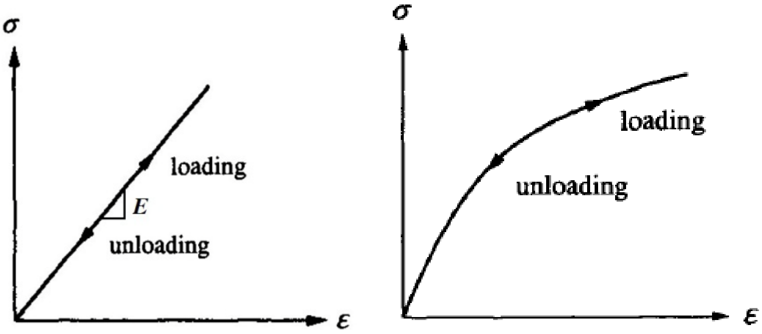

Figure 2 Elastic response of material: (a) linear, (b) nonlinear [5]

Viscoelasticity

A material is said to be viscoelastic if the material has an elastic (reversible) part as well as a viscous (irreversible) part (Figure 3). Upon application of a load, the elastic deformation is instantaneous while the viscous part occurs over time. The viscoelastic model usually depicts the deformation behavior of glass or glass-like materials and may simulate cooling or heating sequences of such material. These materials at high temperature turn into viscous fluids and at low temperature behave as solids. Viscoelastic behavior is also used to model polymeric materials and polymer based composites, as well as biological tissue.

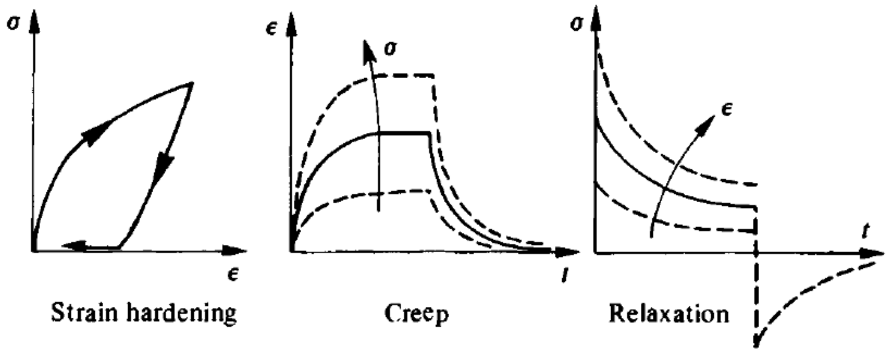

Figure 3 Viscoelastic solids [4]

2.3.2. Plastic solids

Material behavior of plastic solids is characterized by irreversible straining that occurs in a material once a certain level of stress is reached. The plastic strains are assumed to develop instantaneously, that is, independent of time.

Rigid perfectly plastic solid

The simplest plasticity model is a *rigid perfectly plastic solid* (Figure 4). It changes its shape only if loaded above its yield stress, and then deforms at constant stress. This is applicable when modeling the behavior of soil, or in analysis of metal forming.

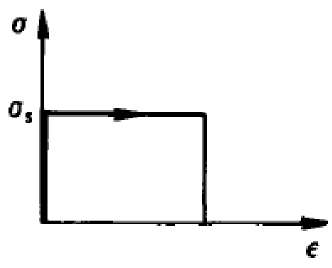

Figure 4 Rigid perfectly plastic solid [4]

Elastic-perfectly plastic solid

An *elastic-perfectly plastic solid* (Figure 5) deforms according to linear elastic equations when loaded below the yield stress, but deforms at constant stress if yield is exceeded. These models would be appropriate to predict energy dissipation in a crash analysis, or to calculate tool forces in a metal cutting operation, for example.

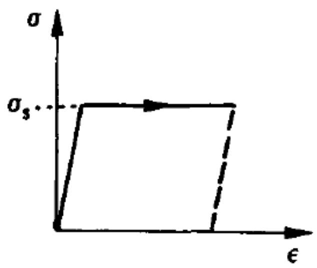

Figure 5 Elastic-perfectly plastic solid [4]

Elastoplastic hardening solid

More sophisticated models, like *elastoplastic hardening solid* (Figure 7) describe *hardening* of the material in some way (the change in the yield stress of the solid with plastic deformation). These are used in modeling ductile fracture, low cycle fatigue (where the material is repeatedly plastically deformed), and when predicting residual stresses and springback in metal forming operations.

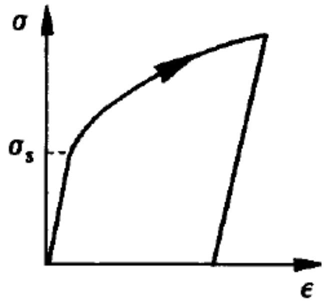

Figure 6 Elastoplastic hardening solid [4]

2.3.3. Viscoplasticity

Viscoplasticity is similar in structure to metal plasticity, but accounts for the tendency of the flow stress of a metal to increase when deformed at high strain rates. Viscoplastic constitutive equations are also used to model *creep* – the steady accumulation of plastic strain in a metal when loaded below its yield stress, and subjected to high temperatures (30-60% of the material melting temperature).

3. PLASTIC BEHAVIOR OF MATERIALS

It has been said, in previous chapter, that material behavior of plastic solids is characterized by irreversible straining that occurs in a material once a certain level of stress is reached. That level of stress is called **the point of yielding**. The plastic strains are considered to be time-independent as long as the time-dependent (creep) deformations, that are always present, are relatively small.

Plastic deformations are of great significance in engineering analysis, since they can significantly impair the functionality of engineering components by causing large deflections or residual stresses which remain in material after unloading, and affect the material resistance to fatigue or environmental cracking. Characterization of plastic deformation behavior of materials, i.e. the nonlinear relationships between stresses and strains during the plastic deformation, is thus a foundation to any engineering analysis.

3.1. HARDENING BEHAVIOR OF MATERIALS

Consider the basic behavior of an elastoplastic material subjected to tension test, as it is shown in Figure 7. In the beginning, the behavior of material is linear elastic with modulus of elasticity E. After the initial yield stress σ_{yo} is reached, plastic strains develop. Unloading from point A occurs elastically with the modulus of elasticity E so that at complete unloading to point B, the residual strain amounts to the plastic strain ε^p developed at point A. Therefore, at point A, the total strain ε consists of the sum of the elastic and plastic strains [5]

$$\varepsilon = \varepsilon^e + \varepsilon^p \qquad (1)$$

If we reload again from point B, the material responds elastically until the stress reaches value σ_y at point A, which is the value of current yield stress. Current yield stress in general differs from the initial yield stress. On loading beyond point A the material behaves as if the previous unloading from point A had never occurred. When yield stress on reloading is greater than the yield stress of a virgin material, we say that material is hardened.

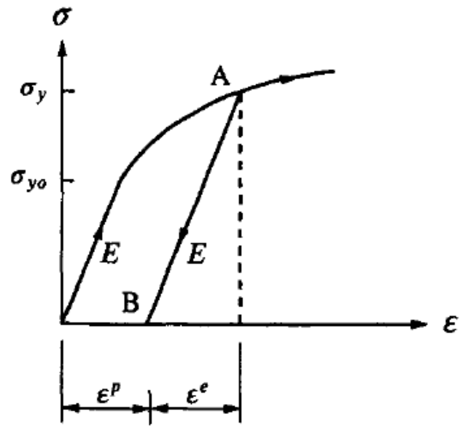

Figure 7 Basic response of elastoplastic material [5]

For most metals and steels, stress-strain curves in tension and compression are nearly identical, as shown in Figure 8.

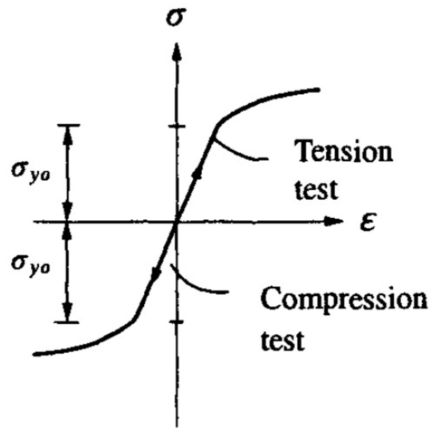

Figure 8 Stress-strain curves for monotonic loading [5]

Now consider the uniaxial loading of material where the tension test is followed by compression test, Figure 9. We expect the unloading curve to follow the stress-strain curve for monotonic loading, but the stress-strain path differs from the monotonic one (dashed line in Figure 9). Yielding on unloading now occurs at point A in Figure 9, that is, prior to the initial yield stress from monotonic compression test σ_{oc}. This early yielding behavior is called the **Bauschinger effect**.

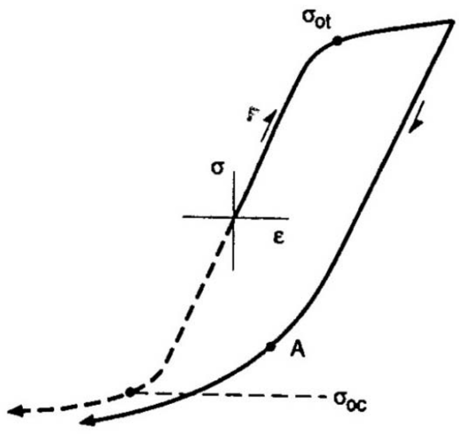

Figure 9 Unloading stress-strain curve: Bauschinger effect [6]

We can say that there are generally two types of hardening behavior of materials [6]. One, based on assumptions from monotonic test, and one based on real material response. The first one is called the **isotropic hardening behavior** and presumes that yielding of material in unloading occurs when the stress range is changed by twice the value of the highest stress reached prior to unloading ($\Delta\sigma = 2\sigma'$), as shown in Figure 10. The name for *isotropic* hardening derives exactly from the assumption that material hardens uniformly in every direction.

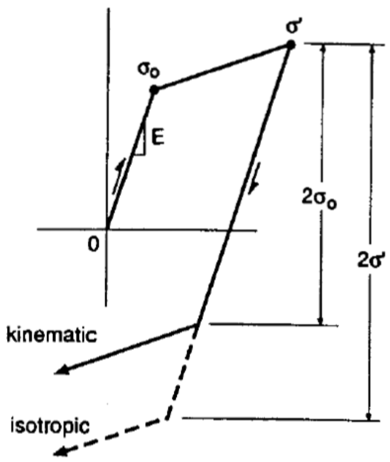

Figure 10 Different unloading behavior for kinematic and isotropic hardening [6]

The second hardening behavior observed is the **kinematic hardening**. Kinematic hardening predicts that the yielding in the reverse direction occurs when the stress change from the unloading point is twice the monotonic yield stress ($\Delta\sigma = 2\sigma_o$), Figure 10. Kinematic hardening behavior thus accounts for the phenomenon of Bauschinger effect (early yielding in compression) and is often more suitable than isotropic hardening for representing the real material behavior for cyclically loaded materials.

3.2. EFFECT OF CYCLIC LOADINGS

3.2.1. Cyclic hardening/softening behavior and stress-strain curve

Most metals and alloys experience a variation in their hardening properties during the tension-compression cyclic loading [4]. Depending on the material, temperature and initial state they may harden or soften.

When the stress range $\Delta\sigma$ decreases during strain controlled cyclic loading, or when the strain range $\Delta\varepsilon$ increases in a stress controlled test, cyclic **softening** of material occurs (Figure 11(a) and (b)).

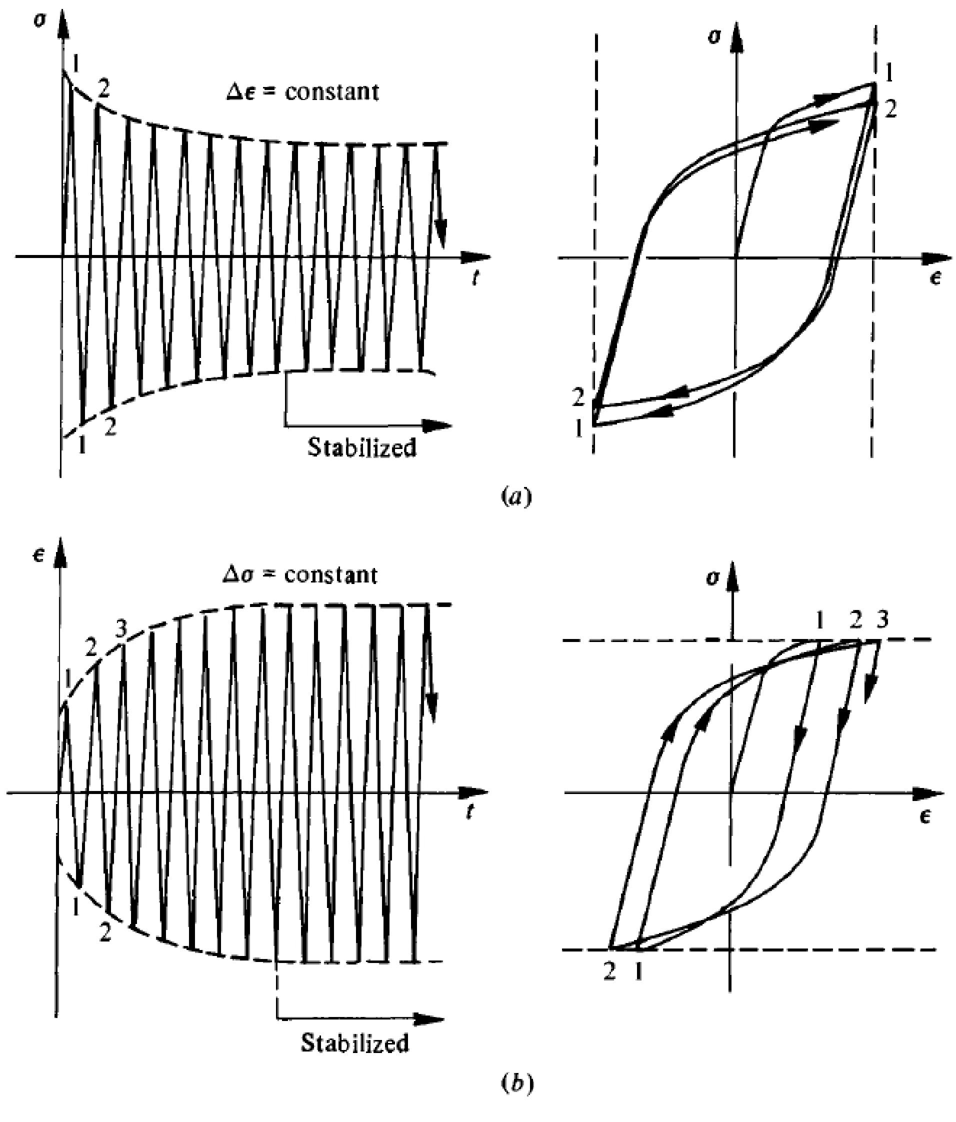

Figure 11 Phenomenon of cyclic softening: (a) strain controlled; (b) stress controlled test [4]

On the other hand, rise in the stress range $\Delta\sigma$ in strain controlled test or a fall in the strain range $\Delta\varepsilon$ when the test is stress-controlled, indicates **cyclic hardening** (Figure 12(a) and (b)).

Both softening and hardening are usually rapid at first, but decrease with number of cycles and the behavior often becomes approximately stable.

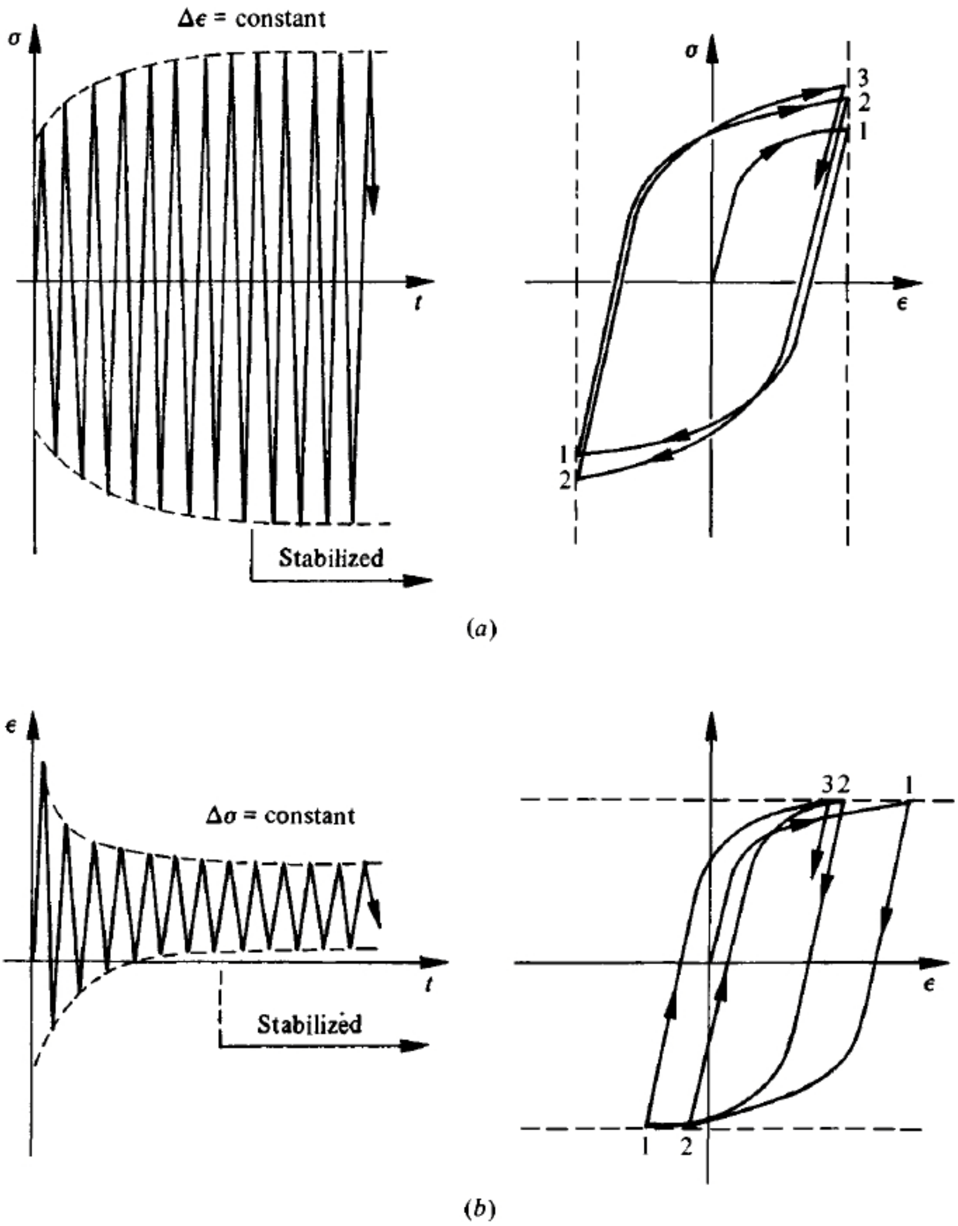

Figure 12 Phenomenon of cyclic hardening: (a) controlled strain; (b) controlled stress [4]

Figure 13 represents the stress-strain variation during stable behavior for one, completely reversed, loading cycle. It can be seen that on each cycle, a **closed hysteresis loop** is formed.

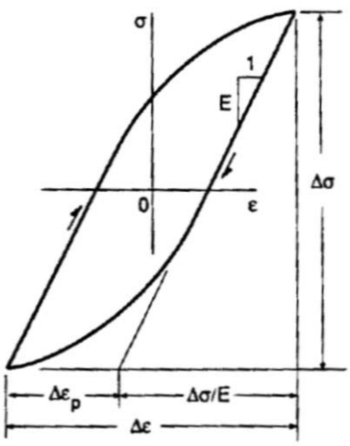

Figure 13 A stress-strain cycle [6]

After the direction of loading changes at the positive or the negative strain limit, the slope of the stress-strain path, after a first constant slope (close to elastic modulus *E* from tension test), gradually deviates from linearity as plastic strain occurs.

Conventionally, to represent approximately stable behavior, hysteresis loops from near half the fatigue life are used. Figure 14 represents a plot of a series of such loops from tests at several different strain amplitudes. The line that passes through origin and the tips of the loops is called **cyclic stress-strain curve.**

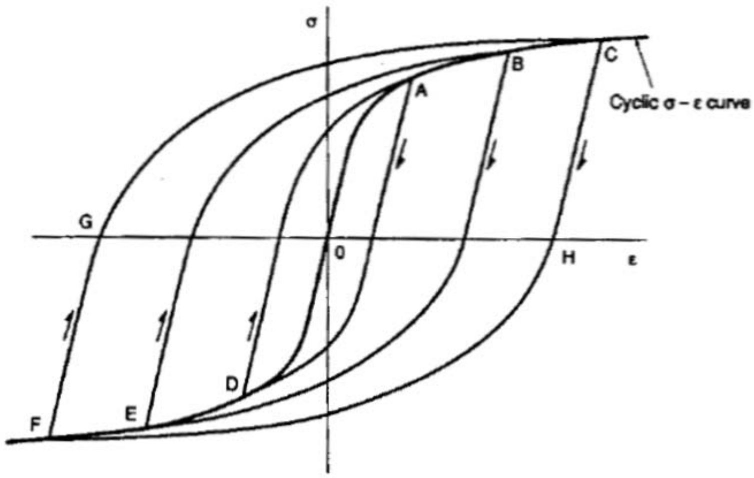

Figure 14 Cyclic stress-strain curve defined as the locus of tips of hysteresis loops [6]

The relationship between stress and strain amplitudes for cyclic loading, i.e. cyclic stress-strain curve is commonly represented with **Ramberg-Osgood equation** [6]:

$$\frac{\Delta\varepsilon}{2} = \frac{\Delta\sigma}{2E} + \left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}} \tag{2}$$

where:

E - Young's modulus
$\Delta\varepsilon/2$ - strain amplitude
$\Delta\sigma/2$ - stress amplitude
$K^{'}$ - cyclic deformation hardening coefficient
$n^{'}$ - cyclic deformation hardening exponent.

Since Ramberg-Osgood cyclic equation has rather simple form, material parameters can quite accurately be estimated on the basis of parameters obtained from simple, fast and relatively inexpensive monotonic tests.

Generally, if the cyclic stress-strain curve lies above than monotonic curve, we say that material has hardened, and if it lies below we say it has softened. Mixed behavior can also occur.

3.2.2. Cyclic behavior in non-symmetric tests

Previously described behavior is valid for purely alternating load. If the mean load is not zero, additional effects can occur [1], [2], [4]. Typical behavior for tests conducted with non-fully reversed cyclic loading is the shifting of hysteresis loops along the corresponding axis.

If biased stress limits are imposed in stress-controlled tests, either shakedown or more often ratcheting, can occur. **Shakedown** (Figure 15(a)) occurs when moving hysteresis loops accommodate at a certain level of deformation. Shakedown is a desirable state because it means that the structure is essentially elastic after some initial plasticity during first few load cycles. When repeated loading cycles do not produce a repeating hysteresis loop after a few cycles, the loop usually progresses steadily to the right. With every load cycle more plastic deformation is accumulated. Such behavior is called **ratcheting** or **cycle-dependent creep** (Figure 15(b)).

In a strain-controlled test the phenomena of **relaxation** or **nonrelaxation** of the mean stress (Figure 15(c) and (d)) occur. If biased strain limits are imposed, and the strain range is large enough to cause some plasticity, the resulting mean stress will gradually shift towards zero value. Mean stress will reach zero only if the degree of plasticity is large, otherwise it will achieve a stable value of mean stress, lower than the initial one.

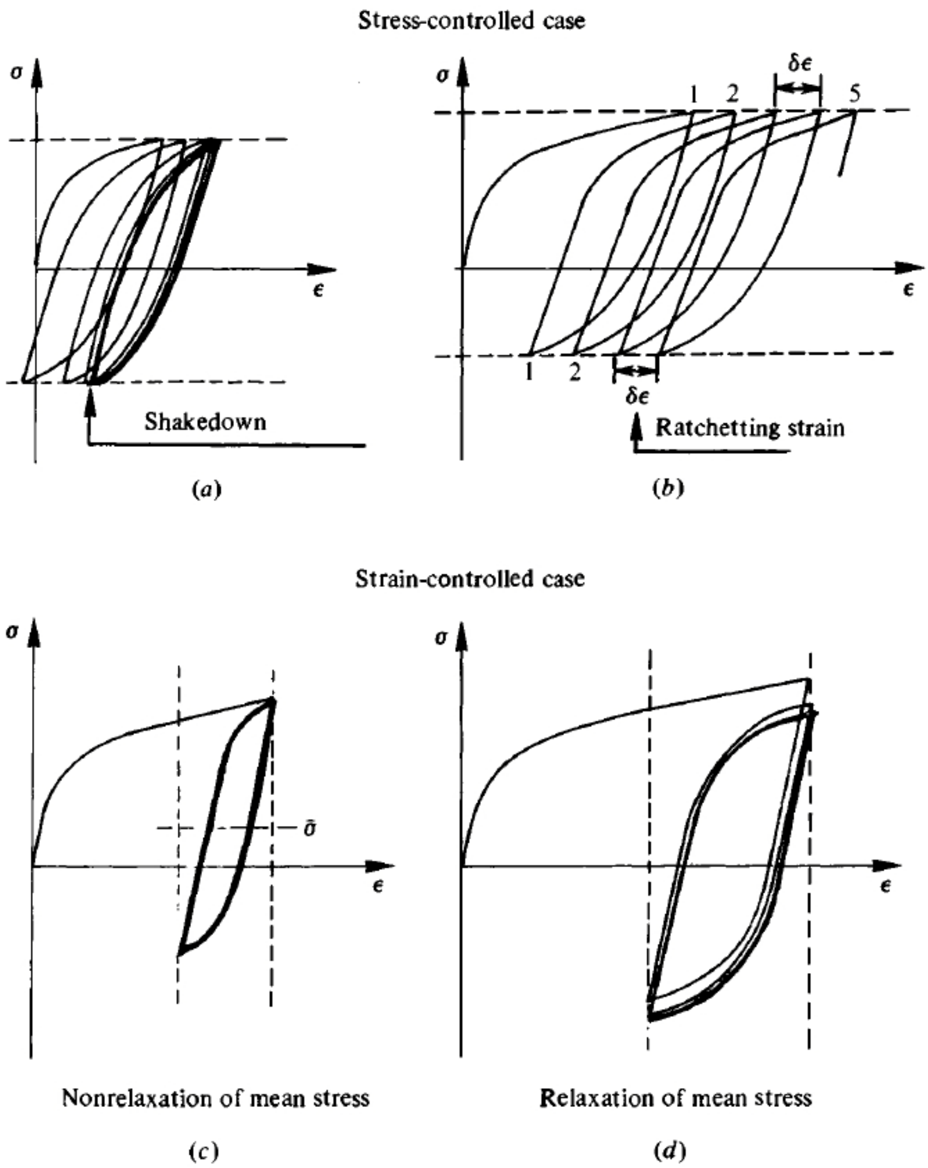

Figure 15 Phenomena of (a) shakedown, (b) ratcheting, (c) non-relaxation and (d) relaxation of the mean stress [4]

4. DAMAGE OF MATERIALS

The physical process that occurs concurrently with deformation of material and leads to failure of material is called **the damage of materials. The damage mechanics** is the study of successive degradation phenomena of the material subjected to loading, from initial (undamaged or predamaged) state up to creation of the crack. There are three different size scales of interest when considering the damage of materials [6]:

- the **microscale**: damage is considered as an accumulation of microstresses in the neighborhood of defects or interfaces and the breaking of bonds, which both damage the material
- the **mesoscale** (level of the RVE): damage is considered as growth and coalescence of microcracks and microvoids which initiate the crack together
- the **macroscale**: damage is considered as discontinuity on the macro scale that leads to global failure (fracture).

The first two stages are usually studied by means of damage variables of the mechanics of continuous media (defined at mesoscale level) while the third stage is usually studied using fracture mechanics (variables are defined at macroscale level).

Both deformation and damage of materials depend significantly on microstructure of material (already discussed in chapter 2.1), so it is very valuable to be familiar with the mechanisms that occur on the microlevel (of atoms or crystal grains) for better understanding of the observed, (macrolevel) behavior.

4.1. DEFORMATION AND DAMAGE OF MATERIALS

Elastic deformation

All materials are composed of atoms held together by chemical bonds. Elasticity is related to the relative movement of atoms, i.e. stretching but not breaking the chemical bonds. When an external stress is applied to a material, the distances between the atoms change by a small amount (depending on its structure and bonding). Sum of these distance changes over a piece of material of macroscopic size is called the **elastic deformation**. Once the stress is removed, the elastic deformation disappears, i.e. we say it is **reversible**.

Plastic deformation

More drastic events occur when after the applied stress, the atoms rearrange so that they have new neighbors after the deformation is complete. Such deformation is permanent and does not disappear when the stress is removed. This **irreversible** deformation is called **plastic deformation.** Crystalline materials, such as metals and ceramics, are composed of aggregations of small grains, each of which are an individual crystal, and separated by grain boundaries. Within grains, the crystals include imperfections, which can be classified as *point*, *line* or *surface defects* and together with the grain boundaries can have large impact on material behavior.

Line defects are called **dislocations** and they are primarily responsible for lowering the strength of metals and causing the yielding behavior. When shear stress is applied, an incremental process of moving the dislocations occurs, causing atoms to position in a stable configuration after the dislocation has passed. Movement of the dislocations causes slips, i.e. the plastic strain (Figure 16). Slips can also occur by climbing of dislocations and twinning. No debonding occurs.

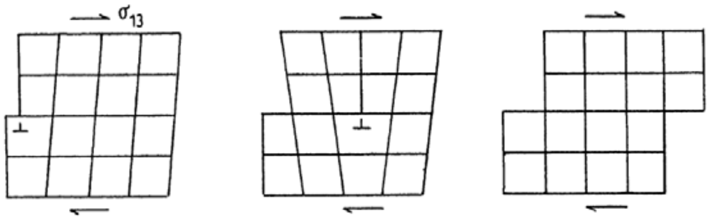

Figure 16 Elementary plastic strain by slip due to dislocation movement [7]

Damage of materials

Dislocation movement can be stopped by a microdefect or a microstress concentration in a crystal lattice, forming a constrained zone that can stop another dislocation from moving. If this happens, plastic deformation cannot continue without breaking the atomic bonds, thus resulting in damage [1], [4], [6]. Microcrack created in such way is shown in Figure 17.

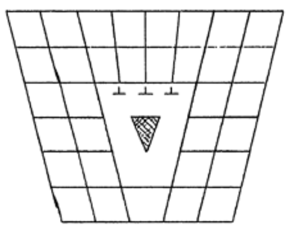

Figure 17 Elementary damage by nucleation of a microcrack due to an accumulation of dislocations [7]

Damage in metals can occur also due to intergranular debonding or decohesion between inclusions and the matrix. No matter of the damage mechanism, plastic microstrains occur.

Number of atomic bonds is the measure of elastic stiffness. Since it decreases with damage, elastic stiffness (modulus of elasticity) also decreases with damage. Thus we can say that damage has the direct influence on material elasticity.

Plastic strains are also influenced by damage since the elementary area of resistance decreases as the size of damage increases. It must be emphasized that development of damage in metals is not the same thing as deformation, but it is coupled to development of the plastic strain. The real measure of damage is the reduced elasticity modulus.

4.2. DIFFERENT MANIFESTATIONS OF DAMAGE

Although the damage at the microscale level is driven by one of the debonding mechanisms, at mesoscale level it occurs in various ways considering the nature of the materials, type of loading, and the temperature.

Brittle damage

The cracking that occurs in a material without previous large amount of plastic strain is called **brittle** damage **(**Figure 18**)**. Brittle damage develops with the total strain when the cleavage forces become higher than the debonding forces and a certain threshold of strain has been exceeded. The degree of localization is high.

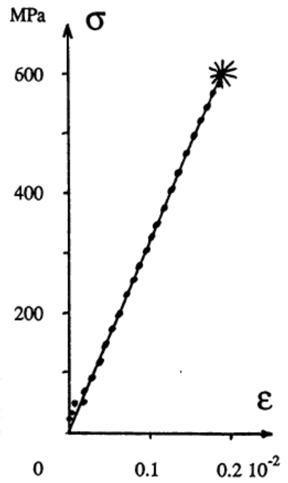

Figure 18 Tensile stress-strain curve of the brittle material up to rupture [7]

Ductile damage

Cracking that occurs simultaneously with plastic deformations larger than a certain plastic strain threshold is called the **ductile** damage. Due to severe stress concentrations in the neighborhood of dislocation obstacles, cavities nucleate in the material. Nucleation phase is followed by the growth and coalescence of these cavities which leads to the phenomenon of the plastic instability of material. The degree of localization of ductile damage is comparable to that of plastic strain. Tension stress-strain curve and the ductile damage manifestation of ductile steel are given in Figure 19.

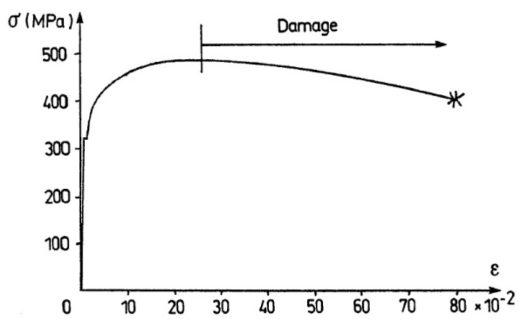

Figure 19 Tension stress-strain curve of ductile steel [7]

Creep damage

Creep damage is a time-dependent, thermally activated process of deterioration of material properties when loaded at elevated temperatures (for metals approximately above 30% of melting temperature). The plastic strain involves viscous part – material may be deformed even at constant stress (Figure 20). Creep damage is usually associated with tertiary creep – when the strain is large enough, intergranular decohesions occur, thus producing damage. Area of the load bearing cross-section reduces. Local mean stresses rise, and so does the strain rate. Overall strain increases due to voids and microcracks, and such unsteady, accelerating process ends with fracture.

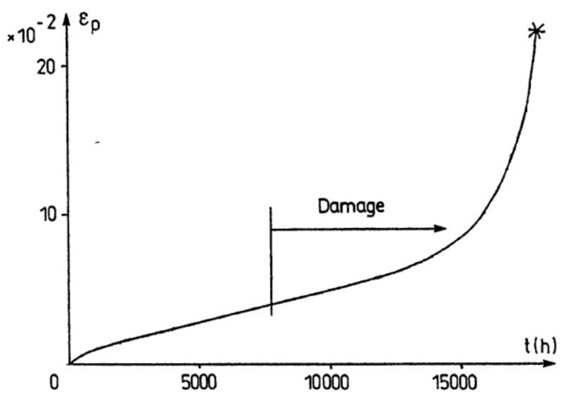

Figure 20 Creep curve under constant tension stress of A 542 stainless steel at 550°C [7]

Low cycle fatigue damage

When material is subjected to cyclic loading with high values of stress or strain, global plasticity occurs and develops with every cycle. After initial phase of plastic strain accumulation, when material hardens or softens, and achievement of the stable behavior (stable hysteresis loop), damage, i.e. nucleation and propagation of microcracks, starts to develop along with cyclic plastic strain. This damage development will eventually result in cyclic softening all the way until fracture – elastic unloading modulus will continually decrease, as will the stress amplitude in strain-controlled test (Figure 21, two hysteresis loops represent the stabilized cycle and a cycle close to the rupture). In stress-controlled test strain amplitude grows in an uncontrolled fashion. The number of cycles to failure is relatively small, $N_R < 10000$ cycles and final fracture is ductile.

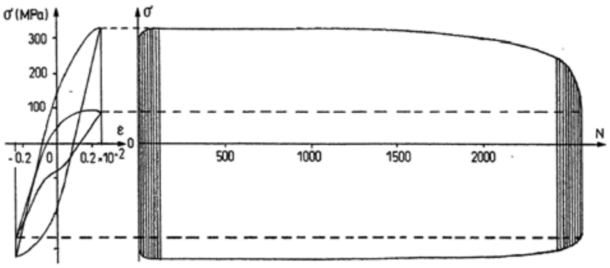

Figure 21 Cyclic tension compresion curves for low cycle fatigue of A 316 L stainless steel [7]

High cycle fatigue damage

When a material is subjected to a cyclic loading with stresses lower than macroscopic yield strength, strains on macrolevel remain in elastic range (plastic strains are usually enough small to be neglected). Regardless of the overall elastic response, it is observable that after many load cycles elastic modulus reduces. Now let's recall that lowering of elasticity modulus is directly related to damaging of materials. What happens on the microlevel, and causes the degradation of elastic stiffness, is that the plastic deformation is localized in regions of stress concentration so that microcracking and microslip occur. The damage is thus highly localized. The number of cycles to failure is very high ($N_R > 100000$ cycles), and the final fracture manifests itself in brittle manner since there is virtually no plastic deformation preceding the failure.

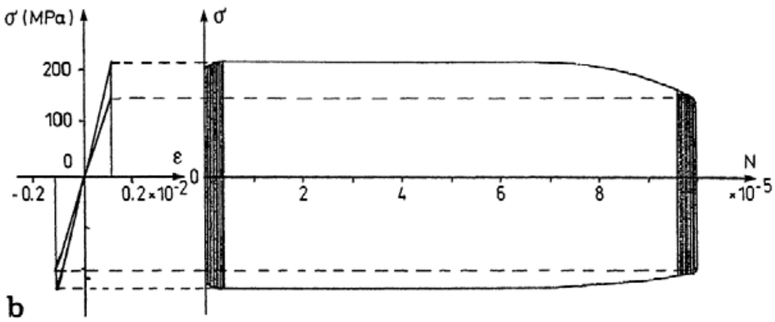

Figure 22 Cyclic tension compression curves for high cycle fatigue of A 316 stainless steel [7]

5. MODELING OF ELASTOPLASTIC MATERIAL BEHAVIOR

Mathematical relationships that characterize the elastoplastic response of materials are provided by **plasticity theory**. Modeling of plasticity is based on three main ingredients (i.e. theoretical expressions): the **yield criterion**, **flow rule** and the **hardening rule** [8], [9], [10].

5.1. YIELD CRITERION

The yield criterion determines the stress level at which yielding is initiated. Plastic behavior of homogeneous, isotropic materials can be represented by a yield function:

$$F(\sigma_{ij}) = f(\sigma_{ij}) - \sigma_y \tag{3}$$

where:

$f(\sigma_{ij}) = \sigma_e$ – equivalent stress, depends on the stress tensor
σ_y – material yield parameter.

Yield function actually describes the surface in the stress space that demarks the boundary between the elastic and plastic behavior of materials, called the **yield surface.** Figure 23 depicts yield surfaces for some of the plasticity options.

Three possible cases of stress state can occur:

- $F(\sigma_{ij}) < 0$, i. e. the equivalent stress is lower than material yield $f(\sigma_{ij}) < \sigma_y$. The stress state point is inside the yield surface and material behaves elastically (no plastic strains occur).
- $F(\sigma_{ij}) = 0$. **Plasticity condition** – the equivalent stress is equal to material yield $f(\sigma_{ij}) = \sigma_y$. Stress state point lies on the yield surface.
- $F(\sigma_{ij}) > 0$, plastic behavior of material occurs.

The third option is actually not a possibility – equivalent stress can never exceed the material yield because plastic strains would in that case develop instantaneously in order to reduce the stress to material yield. Thus, stress state point can only lie **on** the yield surface, so if the stress state tends to move outside the surface, the yield surface "moves" along. This behavior is described by **hardening rule**, Chapter 5.3.

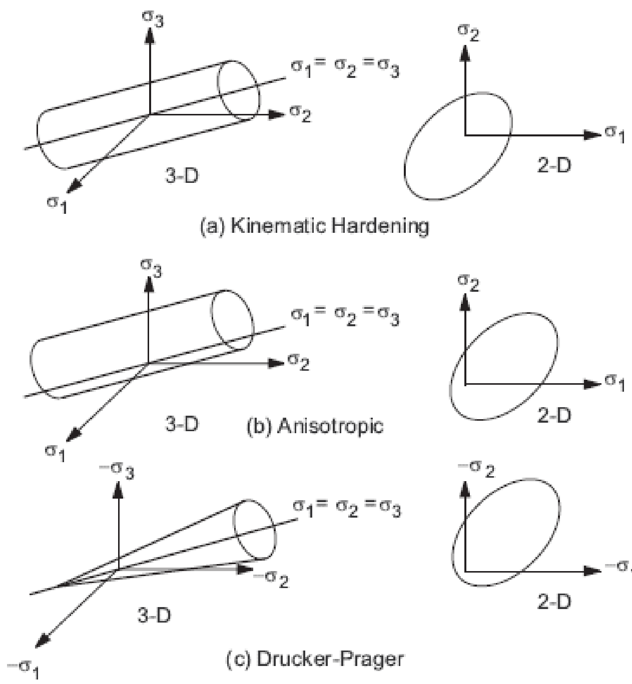

Figure 23 Various yield surfaces [8]

5.2. FLOW RULE

The flow rule determines the evolution (the direction) of plastic straining. It is given by:

$$d\varepsilon_{ij}^{pl} = \lambda \frac{\partial Q}{\partial \sigma_{ij}} \quad (4)$$

where:

λ = **plastic multiplier** (determines the amount of plastic straining), $\lambda > 0$
Q = function of stress termed the **plastic potential**.

Plastic straining is thus proportional to stress gradient of the plastic potential.

If the plastic potential and the yield function are equal ($Q = F$) the flow rule is called **associative.** This is valid for stable materials for which, according to Drucker's postulate, the yield surface is convex (*convexity rule*) and vector of the plastic strain increment in a smooth point of the yield surface is directed along outward normal of the yield surface (*normality rule*). This ensures uniqueness of the boundary value problem solution.

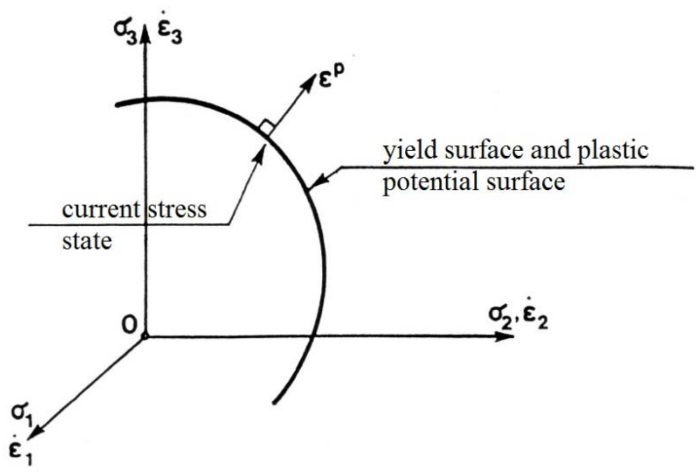

Figure 24 Associative flow rule [9]

Non-associative flow rule occurs when the plastic potential is not equal to yield function ($Q \neq F$) and describes the softening behavior of unstable materials.

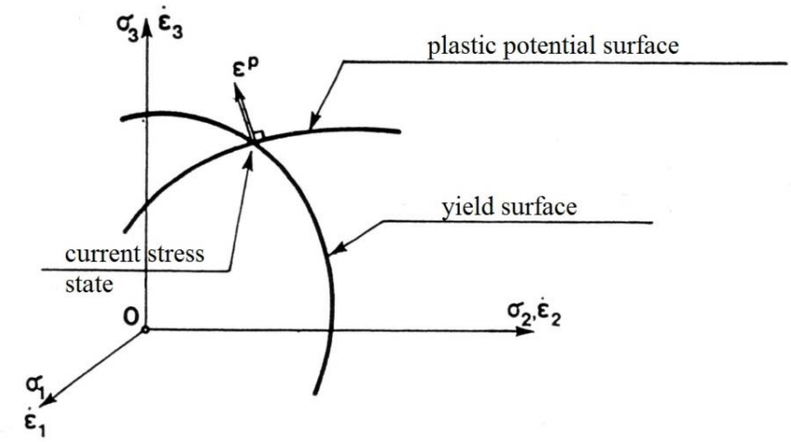

Figure 25 Non-associative flow rule [9]

5.3. HARDENING RULE

Hardening rule represents the condition for establishing the subsequent yielding behavior once the initial yielding has occured. Given that a point in stress space cannot lie outside of the yield surface, it follows that the yield surface has to change its size or position (even shape) for further yielding to occur.

We differentiate two basic hardening rules:

- **isotropic hardening rule**
- **kinematic hardening rule.**

Isotropic hardening rule presumes that the yield surface remains centered about its centerline ($\sigma_1 = \sigma_2 = \sigma_3$) in stress space and expands or shrinks in size as plastic strains develop (depending on hardening or softening behavior, respectively) (Figure 26(a)).

Kinematic hardening rule presumes the translation of the yield surface through the stress space with the progressive yielding (Figure 26(b)). Yield surface remains constant in size.

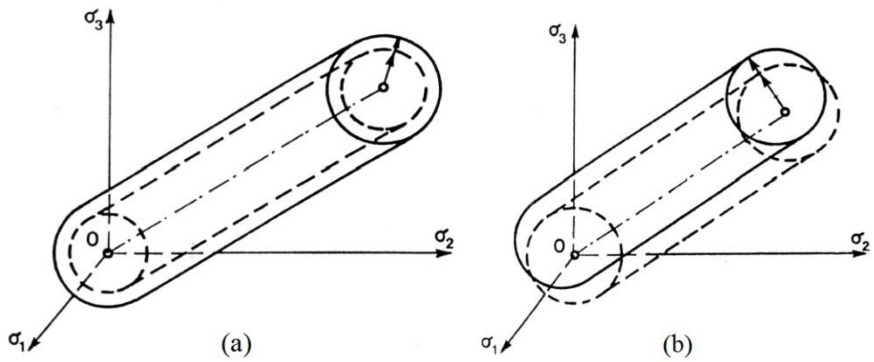

Figure 26 Types of hardening rules: (a) isotropic hardening, (b) kinematic hardening [9]

Mixed hardening behavior (combination of isotropic and kinematic hardening) and **distortional hardening behavior** (where yield surface changes its shape) can also occur, but those exceed the scope of this paper.

6. NONLINEAR PLASTICITY MODELS INCLUDED IN ANSYS

6.1. FEA AND ANSYS

Product development or product enhancement nowadays requires high efficiency, low money consumption and fast delivery of the results. In order to obtain the behavior of an actual engineering system, advanced computational possibilities available today are used. Those allow us to perform simulations and analyses of an engineering component or system, without having to manufacture it (which saves time, money and offers an easier way for possible changes). Powerful tools for performing these are finite element analyses (FEA) software packages, among them, as one of the most popular, ANSYS. The main goal of every FEA is to re-create mathematically the response of an actual structure or component to real loading conditions.

There are three basic steps of FEA: building the model, applying loads and obtaining the solutions, and finally, reviewing of results. Building the model, except defining the geometry, requires the use of mesh generation techniques for dividing complex problems into small elements. Properties, such as material properties that correspond to certain material behavior, of these elements must be defined in order to perform simulations and analyses. It has been mentioned in chapter 2.1 that material behavior choice depends not only on the material itself, but also the purpose and the required accuracy of analyses, next chapter will give an overview of nonlinear material models available in ANSYS software with emphasis on plasticity options, without the detailed discussion on their usage.

6.2. OVERVIEW OF NONLINEAR PLASTICITY MODELS IN ANSYS

In general, material nonlinearities occur because of the nonlinear relationship between stress and strain. This relationship is, except for the case of nonlinear elasticity and hyperelasticity, path-dependent, thus the stress depends on the strain history as well as the strain itself. There are many nonlinear material models included in ANSYS, generally divided as in Figure 27.

Software also accounts the material models for specialized materials or loading conditions, such as those for gasket materials or swelling (material enlargement caused by neutron flux or other effects), but those are not of interest in this paper.

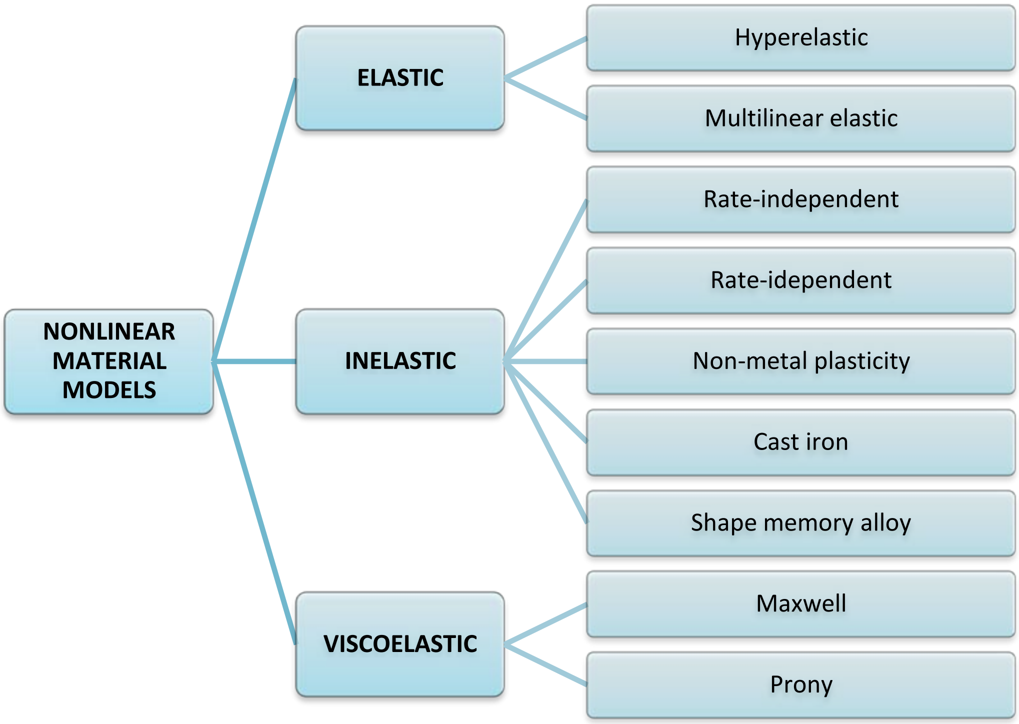

Figure 27 Basic classification of nonlinear material models in ANSYS

The emphasis of this paper is on the rate-independent plastic behavior of metallic materials. Figure 28 represents the basic classification of rate-independent plasticity options in ANSYS.

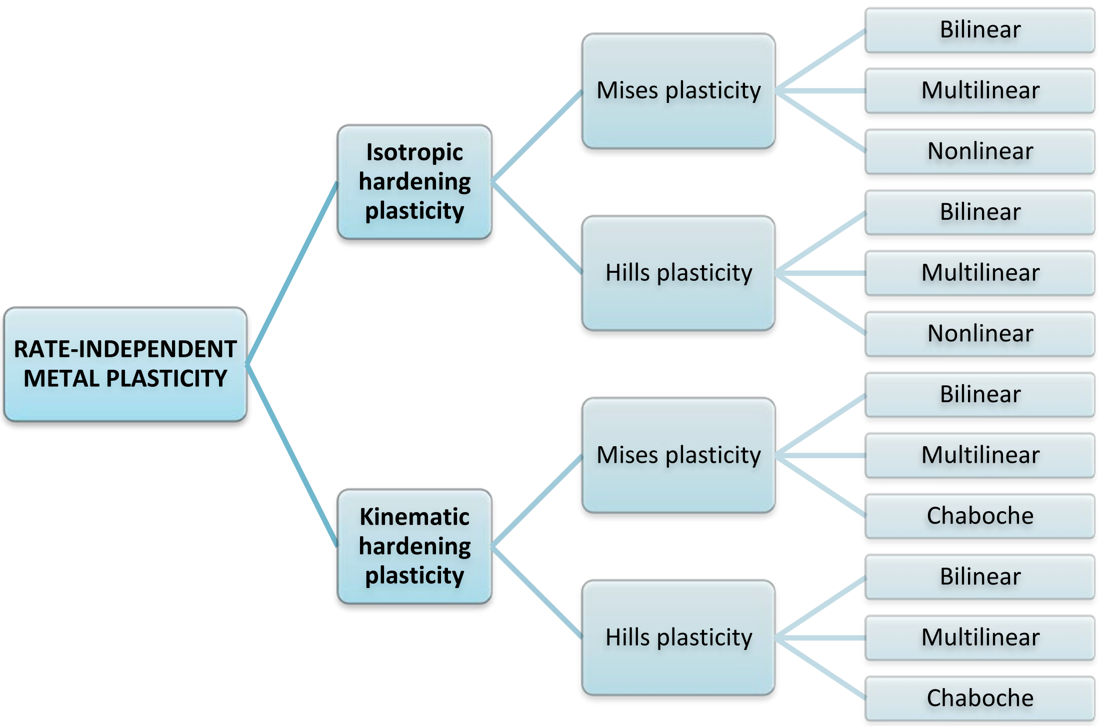

Figure 28 Basic classification of rate-independent plasticity models in ANSYS

Since metallic materials of interest are considered to be **homogeneous** (material properties are equal at all points within the solid) and **isotropic** (material properties are the same in all directions), options that include Hill's plasticity (anisotropic behavior of materials) will not be considered. Uniaxial stress-strain behavior of some of the plasticity options is given in Figure 29.

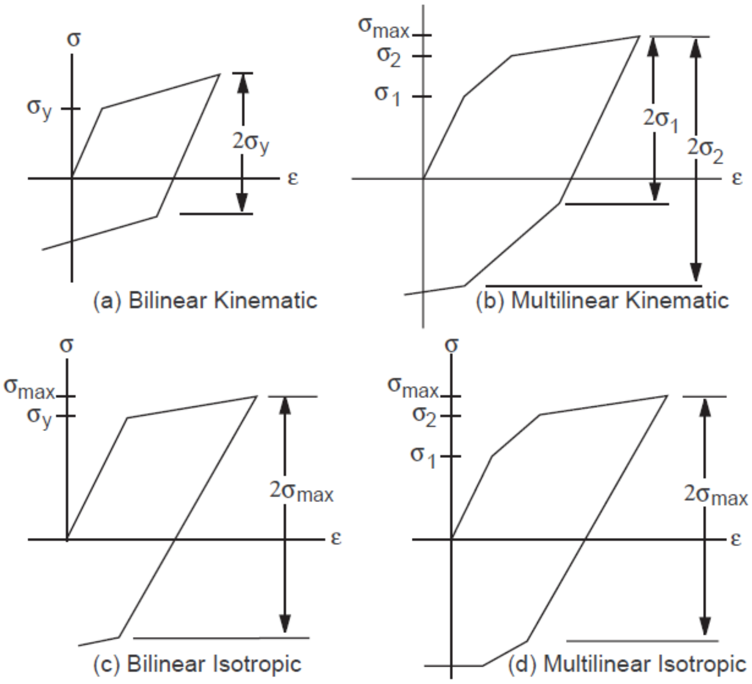

Figure 29 Uniaxial stress-strain behavior of some of the plasticity options [8]

6.2.1. Isotropic hardening plasticity models

6.2.1.1. Multilinear and bilinear isotropic hardening models

Multilinear isotropic hardening behavior ([8], [11], [12]) in uniaxial case of loading is given by a piece-wise total stress-total strain curve, that starts from the origin and has positive values of stress and strain (Figure 29 (d)). Yielding in compression is considered to occur after the stress changes by twice the maximum stress reached in tension.

Multilinear isotropic hardening model uses the von Mises yield criterion with associative flow rule and isotropic hardening.

According to von Mises, the material is assumed to yield when the equivalent stress σ_e is equal to the current yield stress σ_k.

The equivalent stress is:

$$\sigma_e = \sqrt{\frac{3}{2} s_{ij} s_{ij}} \tag{5}$$

where s_{ij} is the deviatoric stress tensor. The von Mises equivalent stress for metallic materials is defined through deviatoric stress (equation (6)) since experimental observations show that hydrostatic stress has negligible effect on the plastic deformation.

$$s_{ij} = \sigma_{ij} - p_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} \tag{6}$$

where:

$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$ is mean or hydrostatic stress

$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$ is Kronecker delta, equal to the unit matrix.

The yield criterion is thus:

$$F = \sqrt{\frac{3}{2} s_{ij} s_{ij}} - \sigma_k = 0 \tag{7}$$

For isotropic hardening, current yield stress σ_k is a function of the amount of plastic work done. For the case of isotropic plasticity assumed here, σ_k can be determined directly from the equivalent plastic strain $\hat{\varepsilon}^{pl}$ and the uniaxial stress-strain curve as depicted in Figure 30.

Current yield stress and equivalent plastic strain are related through plasticity modulus E_P:

$$\sigma_k = E_P \hat{\varepsilon}^{pl} \tag{8}$$

where

$E_P = \frac{E E_T}{E - E_T}$ is the plasticity modulus

E is Young's modulus or modulus of elasticity

E_T is tangent modulus from the uniaxial stress-strain curve.

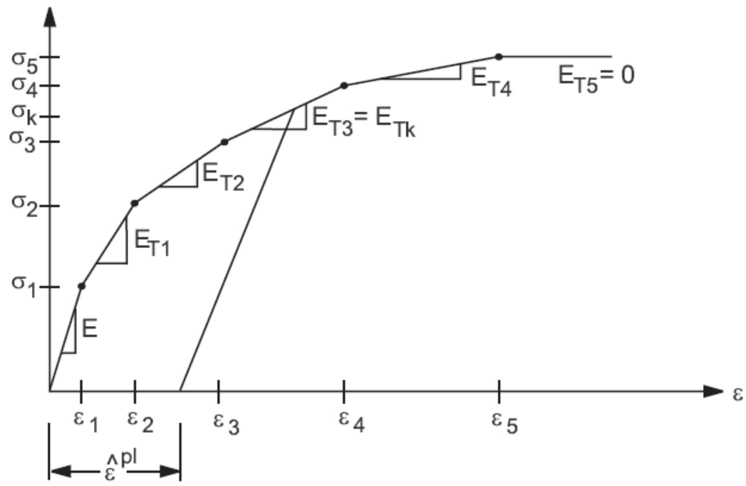

Figure 30 Uniaxial behavior for multilinear isotropic hardening and σ_k determination [8]

Bilinear isotropic hardening option is the same as multilinear isotropic hardening, except that the bilinear curve is used (Figure 29(c)).

These options are recommended for large strain analysis, since for large strain (>5-10% true strain) cycling the kinematic hardening could exaggerate the Bauschinger effect. Their application, though, is not recommended for cyclic or highly nonproportional load histories in small strain analyses.

Both options require prior input of elastic material properties (Young's modulus E, that defines the slope of the initial segment of the curve, and Poisson's coefficient ν). Bilinear option requires the input of two constants, the yield stress σ_y and tangent modulus E_T. At the specified yield stress, the curve continues along the second slope of tangent modulus. The tangent modulus cannot be less than zero (perfect plasticity) nor greater than Young's modulus.

Multilinear plasticity option requires input of stress-strain points on the curve (up to 100 points). Depending on option, strain can be implemented as total strain or plastic strain. Up to 20 temperature dependent input curves can be defined.

6.2.1.2. Nonlinear isotropic hardening model

Nonlinear isotropic hardening model ([8], [10], [11], [12]) assumes that after the initial, linear isotropic elasticity of material, the behavior continues in nonlinear manner. Two hardening laws are available for this option, the Voce hardening law, and the nonlinear power hardening law.

The Voce hardening law for nonlinear isotropic hardening behavior is given in exponential form. Voce hardening option is a variation of bilinear isotropic hardening where an exponential saturation hardening term is added to the linear term (based on assumption that hardening eventually reaches the maximum stress):

$$R = k + R_o\hat{\varepsilon}^{pl} + R_\infty(1 - e^{-b\hat{\varepsilon}^{pl}}) \tag{9}$$

where:

k = elastic limit
R_o, R_∞, b = material parameters characterizing the isotropic hardening behavior of materials where R_o is the initial value of isotropic hardening, R_∞ is the asymptotic value of isotropic hardening (matching the stable cycle regime), while b describes the rate of the cycle stabilization.
$\hat{\varepsilon}^{pl}$ = equivalent plastic strain.

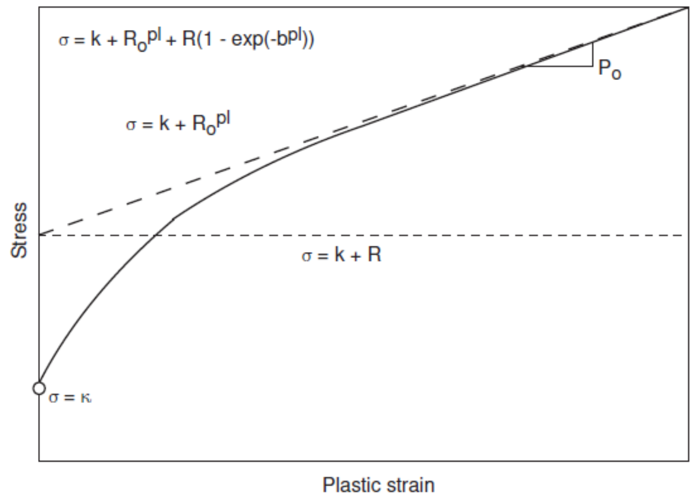

Figure 31 Nonlinear isotropic hardening (Voce hardening) stress-strain curve [12]

The constitutive equations are based on linear isotropic elasticity, the von Mises yield function and the associative flow rule. The yield function is:

$$F = \sqrt{\frac{3}{2}s_{ij}s_{ij}} - R = 0 \tag{10}$$

The plastic strain increment is:

$$d\varepsilon_{ij}^{pl} = \lambda\frac{\partial Q}{\partial\sigma_{ij}} = \lambda\frac{\partial F}{\partial\sigma_{ij}} = \frac{3}{2}\lambda\frac{s_{ij}}{\sigma_e} \tag{11}$$

The equivalent plastic strain increment is:

$$\Delta\hat{\varepsilon}^{pl} = \sqrt{\frac{2}{3}\Delta\varepsilon_{ij}^{pl}\Delta\varepsilon_{ij}^{pl}} = \lambda \tag{12}$$

The accumulated equivalent plastic strain is:

$$\varepsilon^{pl} = \sum \Delta\hat{\varepsilon}^{pl} \tag{13}$$

The second option for nonlinear isotropic hardening is the one that uses power hardening law.

$$\frac{\sigma_Y}{\sigma_0} = \left(\frac{\sigma_Y}{\sigma_0} + \frac{3G}{\sigma_0}\bar{\varepsilon}^{pl}\right)^N \tag{14}$$

where:

σ_Y = current yield stress
σ_0 = initial yield stress
G = shear modulus
N = power value in the power hardening law.

$\bar{\varepsilon}^{pl}$ is the microscopic equivalent plastic strain and is defined by:

$$\dot{\bar{\varepsilon}}^{pl} = \frac{\sigma_{ij}\dot{\varepsilon}_{ij}^{pl}}{(1-f)\sigma_Y} \tag{15}$$

where:

ε^{pl} = macroscopic plastic strain tensor
$\dot{}$ = rate change of variables
σ_{ij} = Cauchy stress tensor
f = porosity.

The first option requires the input of 4 parameters: the elastic limit k, and three material parameters R_o, R_∞, b = material constants. Power hardening option requires the input of 2 parameters, initial yield stress σ_0 and power hardening value N. For both options stress-strain curves can be defined at up to 20 temperatures. Both options require the prior input of elastic material properties.

Nonlinear isotropic hardening option is especially suitable for large-strain analyses. It is not suitable for predicting cyclic loading behavior correctly. For predicting cyclic hardening/softening behavior, it can be combined with Chaboche nonlinear kinematic hardening option.

Power hardening option is primarily suitable for modeling of ductile plasticity and damage, when combined with Gurson's model.

6.2.2. Kinematic hardening plasticity models

Isotropic hardening behavior of materials is generally not useful when simulating cyclic behavior of materials. It doesn't account for the Bauschinger effect, so it predicts that after a few cycles the solid hardens until it responds elastically. Kinematic hardening law allows the translation of yield surface, without changing its shape or size. Bauschinger effect is explained in chapter 3.1. Yielding in compression is considered to occur after the stress changes by twice the initial yield stress reached in tension (thus causing softening of material in compression). Kinematic hardening behavior is suitable for simulating the cyclic loading behavior of materials.

6.2.2.1. Classical bilinear kinematic hardening model

Bilinear kinematic hardening behavior ([8], [10], [11], [12]) in uniaxial case of loading is given by a bilinear total stress-total strain curve that starts from the origin and has positive values of stress and strain (Figure 29 (a)). The model uses the von Mises yield criterion with associative flow rule and Prager's kinematic hardening rule.

The equivalent stress is therefore

$$\sigma_e = \sqrt{\frac{3}{2}(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij})} \tag{16}$$

where α_{ij} is the yield surface translation vector or the "back stress". Equation (16) depends on the deviatoric stress, so the yielding is independent of hydrostatic stress state. When the current yield stress σ_e is equal to the uniaxial yield stress σ_y, the material is assumed to yield.

The yield criterion is:

$$F = \sqrt{\frac{3}{2}(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij})} - \sigma_y = 0 \tag{17}$$

The associated flow rule yields

$$\frac{\partial Q}{\partial \sigma_{ij}} = \frac{\partial F}{\partial \sigma_{ij}} = \frac{3}{2\sigma_e}\left(s_{ij} - \alpha_{ij}\right) \tag{18}$$

Prager's kinematic hardening rule assumes that during plastic deformation, the back stress increment is in the same direction as plastic strain increment:

$$d\alpha_{ij} = Cd\varepsilon_{ij}^{pl} \tag{19}$$

C is the material constant and can be obtained from the uniaxial stress-strain curve. It's related to plastic modulus as follows

$$C = \frac{2}{3}E_P = \frac{2}{3}\frac{EE_T}{E-E_T} \tag{20}$$

The equivalent plastic strain is dependent on the loading history and is defined to be:

$$\hat{\varepsilon}_h^{pl} = \hat{\varepsilon}_{h-1}^{pl} + \Delta\hat{\varepsilon}^{pl} \tag{21}$$

where:

$\hat{\varepsilon}_h^{pl}$= equivalent plastic strain for this time point
$\hat{\varepsilon}_{h-1}^{pl}$ = equivalent plastic strain from the previous time point

The equivalent stress parameter is defined to be:

$$\hat{\sigma}_e^{pl} = \sigma_y + \frac{EE_T}{E-E_T}\hat{\varepsilon}_h^{pl} \tag{22}$$

If there is no plastic straining ($\hat{\varepsilon}^{pl} = 0$), then the equivalent stress parameter $\hat{\sigma}_e^{pl}$ is equal to the yield stress.

Parameter input is similar to that of bilinear isotropic hardening. This option requires prior input of elastic material properties (Young's modulus E, that defines the slope of the initial segment of the curve, and Poisson's coefficient ν) and two constants, the yield stress σ_y and tangent modulus E_T. At the specified yield stress, the curve continues along the second slope of tangent modulus. The tangent modulus cannot be less than zero (perfect plasticity) nor greater than Young's modulus. These can be input at various temperatures.

Two additional suboptions are available for bilinear kinematic hardening:

- Rice's hardening rule, that takes into account stress relaxation with temperature increase, and
- option with no stress relaxation with an increase in temperature (not recommended for non-isothermal problems).

Bilinear kinematic hardening option is recommended for general small-strain use for materials that obey von Mises yield criteria (most metals), and not recommended for large-strain applications. The true stress-strain slope of most metals usually changes with increase of strain which bilinear model fails to account due to its simple representation.

6.2.2.2. Multilinear kinematic hardening model

Multilinear kinematic hardening option is based on Besseling model (sublayer or overlay model) ([8], [11], [12], [13], [14]). This model assumes that an elementary volume of elastoplastic material is composed of various portions (or subvolumes) with equal modulus of elasticity and common total strain, but different values of yield strength. Each subvolume is assumed to exhibit a simple, perfectly plastic behavior, but when combined, the model can represent complex behavior as multilinear stress-strain curve that exhibits the Bauschinger (kinematic hardening) effect (Figure 29(b) and Figure 32).

The difference between yield stresses of each subvolume produces a gradual change of plastic flow of one element after another, causing residual microstresses after relief (responsible for Bauschinger effect).

The portion of total volume (the weighting factor) and yield stress for each subvolume is determined by matching the material response to the uniaxial stress-strain curve. Material is assumed to be perfectly plastic von Mises material. The weighting factor for subvolume *k* is

$$w_k = \frac{E - E_{Tk}}{E - \frac{1-2\nu}{3} E_{Tk}} - \sum_{n=1}^{k-1} w_n \qquad (23)$$

where w_k is the weighting factor (portion of total volume) for subvolume k and is evaluated sequentially from 1 to the number of subvolumes, E_{Tk} is the slope of the kth segment of the stress-strain curve (see Figure 32) and $\sum w_n$ is the sum of the weighting factors for previously evaluated subvolumes.

The uniaxial behavior is described as piece-wise linear total stress-total strain curve that starts from origin, with positive stress and strain values. The slope of the first segment corresponds to Young's modulus. Tangent moduli of other curve segments cannot be greater than the initial slope, nor smaller than zero. The slope of the curve beyond the last defined stress-strain data point is assumed to be zero (perfect plasticity).

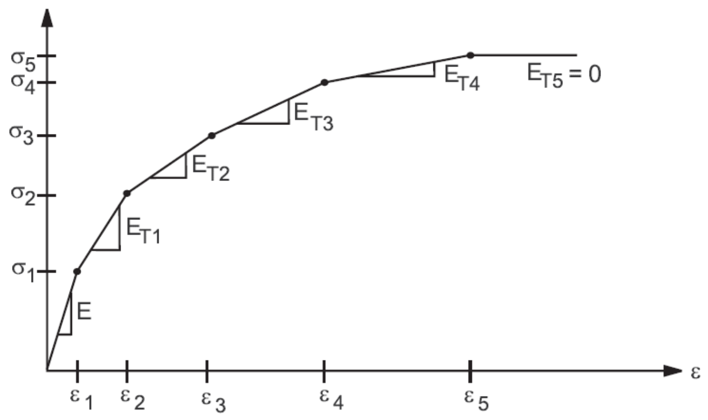

Figure 32 Uniaxial behavior for multilinear kinematic hardening [8]

The yield stress for each subvolume is given by

$$\sigma_{yk} = \frac{1}{2(2+\nu)}\left(3E\varepsilon_k - (1-2\nu)\sigma_k\right) \tag{24}$$

$(\varepsilon_k, \sigma_k)$ is the breakpoint in the stress-strain curve. The number of subvolumes, N_{sv}, corresponds to the number of breakpoints specified.

The increment in plastic strain $\Delta\varepsilon_{ij,k}^{pl}$ for each subvolume is computed using a von Mises yield criterion with the associated flow rule. The specialization for bilinear kinematic hardening (chapter 6.2.2.2) is followed but since each subvolume is elastic-perfectly plastic, C and therefore α_{ij} is zero.

The plastic strain increment for the entire volume is the sum of the subvolume increments:

$$\Delta\varepsilon^{pl} = \sum_{n=1}^{N_{sv}} w_n \Delta\varepsilon_{ij,n}^{pl} \tag{25}$$

The current plastic strain and elastic strain for the entire volume are:

$$\varepsilon_{ij,n}^{pl} = \varepsilon_{ij,n-1}^{pl} + \Delta\varepsilon_{ij}^{pl} \tag{26}$$

$$\varepsilon_{ij}^{el} = \varepsilon_{ij}^{tr} - \Delta\varepsilon_{ij}^{pl} \tag{27}$$

ε_{ij}^{tr} is the trial strain, i.e. the total strain minus the plastic strain from previous time point.

The equivalent plastic strain $\hat{\varepsilon}^{pl}$ is defined by equation (21) and equivalent stress parameter $\hat{\sigma}_e^{pl}$ is computed by evaluating the input stress-strain curve at $\hat{\varepsilon}^{pl}$ (after adjusting the curve for the elastic strain component).

There are two suboptions available for multilinear kinematic hardening option, that differ regarding the input and usage.

The first option (MKIN) has some restrictions:

- up to five temperature dependent stress-strain curves can be defined
- per one curve, only five data points can be used
- each stress-strain curve must have the same set of strain values.

The other option (KINH) allows the user to define stress-strain curves with more number of data points at more temperature levels (if needed):

- up to forty temperature dependent stress-strain curves can be defined
- per one curve, twenty data points can be used (if more stress-strain curves are defined, each curve should contain the same number of points)
- strain can be input as total strain or plastic strain.

These options are not recommended for large strain analyses.

6.2.2.3. Nonlinear kinematic hardening model

Nonlinear kinematic hardening model ([8], [11], [12], [15], [16]) implemented in ANSYS is actually the rate-independent version of Chaboche's nonlinear kinematic hardening model. Constitutive equations are based on linear elasticity, von Mises yield function and associative flow rule. The model can be used to simulate the monotonic hardening and the Bauschinger effect (like bilinear and multilinear kinematic hardening models) but also the ratcheting effect (described in 3.2.2). Model also allows superposition of several kinematic models as well as isotropic hardening models in order to simulate the complicated cyclic plastic behavior of materials, such as cyclic hardening or softening and ratcheting or shakedown.

The model uses the von Mises yield criterion with the associated flow rule. The yield function is:

$$F = \sqrt{\frac{3}{2}(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij})} - R = 0 \qquad (28)$$

where R is isotropic hardening variable.

The flow rule is:

$$\Delta\varepsilon_{ij}^{pl} = \lambda \frac{\partial Q}{\partial \sigma_{ij}} \tag{29}$$

The back stress α_{ij} (yield surface motion through stress space) is superposition of several kinematic models as:

$$\alpha_{ij} = \sum_{k=1}^{n} \alpha_{ij,k} \tag{30}$$

where n is the number of kinematic models to be superposed.

The evolution of the back stress (the kinematic hardening rule) for each component is defined as:

$$\Delta\alpha_{ij,k} = \frac{2}{3} C_k \Delta\varepsilon_{ij}^{pl} - \gamma_k \alpha_{ij,k} \Delta\hat{\varepsilon}^{pl} + \frac{1}{C_k}\frac{dC_k}{d\theta}\Delta\theta\ \alpha_{ij,k} \tag{31}$$

where $\hat{\varepsilon}^{pl}$ is the accumulated plastic strain, θ is temperature, and $C_k, \gamma_k\ (k = 1,2, \dots, n)$ are Chaboche kinematic hardening material parameters for n pairs. These can differ significantly for various kinematic models (factors 5 to 20 between them).

The first term in equation (31) is Prager's linear kinematic hardening rule, describing the evolution of back stress with plastic strain. Since Prager's rule doesn't describe the relationship between back stress and evolution of plastic strain well enough, Armstrong and Frederick added the dynamic recovery term (second term in equation (31)). Gives a recall, or evanescent memory effect and acts with the plastic strain. These two together give Armstrong-Frederick's nonlinear kinematic hardening rule. The third term represents static or time recovery. It is independent of plastic strain and thermally activated.

According to equation (30) Chaboche suggested decomposition of Armstrong-Frederick hardening rule in order to better describe three critical segments of stable hysteresis loop:

1. the initial modulus when yielding starts
2. nonlinear transition after the yielding has begun till the curve becomes linear again
3. linear segment of the curve in larger strains range.

The associated flow rule yields:

$$\frac{\partial Q}{\partial \sigma_{ij}} = \frac{\partial F}{\partial \sigma_{ij}} = \frac{3}{2}\frac{s_{ij} - \alpha_{ij}}{\sigma_e} \tag{32}$$

The plastic strain increment (equation (29)) is then

$$\Delta\varepsilon_{ij}^{pl} = \frac{3}{2}\lambda\frac{s_{ij} - \alpha_{ij}}{\sigma_e} \tag{33}$$

The equivalent plastic strain increment is then:

$$\Delta\hat{\varepsilon}^{pl} = \sqrt{\frac{2}{3}\Delta\varepsilon_{ij}^{pl}\Delta\varepsilon_{ij}^{pl}} = \lambda \tag{34}$$

The accumulated equivalent plastic strain is:

$$\hat{\varepsilon}^{pl} = \sum \Delta\hat{\varepsilon}^{pl} \tag{35}$$

The isotropic hardening variable R, can be defined by Voce hardening law (explained in chapter 6.2.1.2):

$$R = k + R_o\hat{\varepsilon}^{pl} + R_\infty(1 - e^{-b\hat{\varepsilon}^{pl}}) \tag{36}$$

The material hardening behavior R, in equation (28) can also be defined through bilinear or multilinear isotropic hardening options, which have been discussed earlier in chapter 6.2.1.1.

Model requires the input of elastic properties of material. Isotropic hardening variable R is input as the yield stress of material. When the model is combined with one of the isotropic hardening model, R is overridden. Depending on the number of kinematic model requested (n), C_k and γ_k pairs must be inserted. Up to five kinematic models may be superimposed.

7. LITERATURE

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