Approximate solution of eigenvalue problem for modal analysis

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Abstract. The paper presents a novel numerical approach for approximate solution of eigenvalue problem and investigates its suitability for modal analysis of structures. The approach is based on transformation of the matrix equation into frequency domain and subsequent removal of potentially less significant frequencies. The procedure results in a much reduced problem that is used in eigenvalue calculation. After calculation eigenvectors are expanded and transformed back into time domain. The principles are presented in Jeričević [1,2]. The original eigenvalue problem is transformed from the space domain into the frequency domain using the Fourier transformation matrix $F$. If certain portions of $F$ are removed according to established principles in digital signal processing the resulting problem is reduced in size.

In structural analysis boundary conditions are the most important information that characterizes the structure and is not preserved in the pruning process in the frequency domain. In order to keep the boundary conditions present throughout the frequency domain the original method of approximate solution of linear systems using Fourier transforms has been modified.

Numerical examples demonstrate the applicability of the proposed method in modal analysis of structures.

1 Introduction

Solving eigenvalue problem is of fundamental importance in engineering computing. It is also computationally expensive, the cost in number of operations of the order $O(n^3)$ where $n$ is the size of a general, dense square matrix. Practical limits for everyday computing with single processor machines are matrices of the size around 10000 by 10000 while sparse matrices with dimensions of millions are not uncommon. It is of great interest to compute an approximate solution for dense, large matrices. We propose an approach which reduces the size of matrix, but still provides good approximation of full sized problem.

Each frequency in the Fourier domain represents the best approximation in the least squares sense to the original sequence by that particular frequency. Consequently, the approximate solution constructed in the Fourier domain is the least squares approximation of the true solution with the set of chosen frequencies. This can be used to control the accuracy of the approximation. The original (space domain) form of the problem cannot be treated in the same way without perturbing it in an uncontrolled manner.

After transforming to the Fourier domain, the problem of solving the system is still of the same order of complexity. If the linear system started with a real matrix,
the problem becomes more complicated because its transform will generally be complex. The complication of dealing with a set of complex equations can be eliminated by converting our equations into the real Hartley transform domain [3].

2 Linear system in frequency domain

We propose to transform a linear system using the Fourier or Hartley transforms, and in the manner of signal processing discarding the least significant portions of the transformed system in order to solve a reduced size problem in the transform space. This results in approximate transform of solution based on a limited bandwidth of the most important Fourier frequencies. The inverse transformed yields an approximate solution in space domain. The idea was first developed by Jeričević [1,2] for analysis of geophysical signals and some chemical graph theory applications. Original problem in space domain reads

\[ Aq_i = \lambda_i q_i \]  

where \( A \) is a square matrix, \( \lambda_i \) its \( i \)-th eigenvalue and \( q_i \) is the associated \( i \)-th eigenvector. Transformation into the frequency domain is described as follows

\[ (FAF^{-1}) (Fq_i) = \lambda_i (Fq_i) \]  

Matrix \( F^{-1} \) is the inverse of the Fourier matrix \( F \), the matrix product \( AF^{-1} \) is the inverse Fourier transform along the rows of matrix \( A \). An equally valid result would be arrived to by using \( FF^{-1} \) instead of \( F^{-1}F \) in Eq. (2), but \( F^{-1}F \) has the computational advantage in normalization of the Fourier transform.

Above equations could be written in another form

\[ \{A\}C = \{Q\}A \]  

where \( \{Q\}_C \) is a matrix containing Fourier transforms of the eigenvectors (as columns), \( A \) is diagonal matrix of eigenvalues and \( \{A\} \) is the two dimensional Fourier transform of matrix \( A \), taken as the Fourier transform along the columns and the inverse Fourier transform along the rows of matrix \( A \).

The approach outlined above may be considered as a preconditioning of matrix \( A \). That still leaves a problem of the same order of magnitude as the initial linear system problem. However, the problem is now posed in the frequency domain in which the significance of system orthogonal components, the frequencies, is obvious. Consequently, the system can now be easily manipulated by discarding whole rows and corresponding columns from matrix \( \{A\} \).

3 Transformation of dynamic equation into frequency domain

Above procedure will be employed on plate dynamics problem. Discretized version of the plate vibration problem (without damping) is presented in the usual manner

\[ M \ddot{D} + KD = 0 \]  

(4)
Matrix $M$ is the (diagonal) mass matrix and $K$ is the stiffness matrix. Equation (4) is homogenous as it is assumed that the modal superposition method has been chosen for a solution procedure of plate dynamics problem. From this equation we are trying to extract eigenvalues and eigenvectors. In the case of large plate problems time consuming calculations could be expected. Eigenvalue problem in structural dynamics can be written as

$$\left( I - \omega^2 K^{-1} M \right) D = 0$$

(5)

If we compare eq.(5) and (1) it is obvious that $K^{-1} M$ corresponds to matrix A from eq.(1). After that we proceed as described by eq.(2). In this paper size of matrices is reduced prior to eigenvalue problem solution. The method applied is Hartley transform (similar to Fourier but giving real coefficients). Various reductions in size resulting is smaller matrices have been tried out.

Importance of each mode is determined through mass participation ratio (see Wilson [10]) that is described as (squared) ration between modal and kinetic energy.

3.1 Boundary conditions

In this example we have only Dirichlet boundary condition that is the simplest possible case. That type of boundary conditions can be easily introduced (see Trefethen [9]) by excluding points with zero values from Fourier transformation. Since transformed values at those points should be disregarded it is best to reduce the size of input stiffness matrix $A$ prior to transformation. More complicated boundary conditions have to be treated in a different manner.

4 Example problem

In this example plate has been modeled with total of 9341 degrees of freedom that are reduced to 1029 degrees of freedom. Discrete mass matrix has been applied and discrete masses are introduced only into angles of finite elements. In Fig.1 there is graphical representation of stiffness matrix ($1029\times1029$) in space domain and in Fourier domain.

![Graphical representation of stiffness matrix](image)

Figure 1: Graphical representation of stiffness matrix in space and frequency domain
Using Fourier decomposition that number has been further reduced to 980 and 196 degrees of freedom. Periodicity of 49 is clearly visible (in frequency domain) and that is the reason for the size of all reduced matrices to be multiple of 49.

4.1 Comparison of eigenfrequencies and modal shapes

Eigenvalues and eigenvectors are compared for full-size problem (1029 degrees of freedom) and for the reduced problem (in steps of 49 from 1029 to 49 degrees of freedom). The eigenvalues of several modes are compared for all matrix sizes in Fig.2

![Figure 2: Comparison of eigenvalues](image)

Very good agreement is visible up to 5\textsuperscript{th} eigenvalue for all sizes of matrices. After that degree of reduction begins to play important role and great reductions are not possible without significant loss of accuracy (observe the logarithmic scale in ordinate). Comparison of eigenvectors (modal shapes) for mode 1 is presented in Fig.3

![Figure 3: Modal shapes for mode 1 of the original and reduced matrices](image)
Excellent agreement for mode 1 is visible. Comparison of results for mode 7 is presented in Fig.4

![Modal shapes for mode 7 of the original and reduced matrices](image)

**Figure 4: Modal shapes for mode 7 of the original and reduced matrices**
As expected in mode 7 there is not good agreement between original and reduced problem.

### 4.2 Degree of reduction

With excessive reduction of the problem important data could be lost. The following figure demonstrates difference in eigenvector for mode 7 between 980 and 196 degrees of freedom. It is obvious that there is not only quantitative but qualitative disagreement (shapes are completely different).

![Modal shapes for mode 7 of reduced matrices with 980 and 196 degrees of freedom](image)

**Figure 5: Modal shapes for mode 7 of reduced matrices with 980 and 196 degrees of freedom**
At the moment we are working on a simple and efficient procedure that would give the optimal reduction in size.

### 5 Conclusion

The justification for the whole approach lies in constructing the approximate solution, based not on all, but only on the most significant frequencies. This is similar to filtering in digital signal processing, but applied to the linear algebra
problem. The limitation of this filter is in the shape of the filter, which cannot be arbitrary, but has to include whole rows and columns in the frequency domain. By working in the Fourier domain, the size of the matrix will be reduced significantly, if it is feasible. The reduction in size pays off quickly, since the computational problem has an order the size of the matrix dimension to the third power, $O(n^3)$.

The approach outlined above discards whole rows and/or columns from transformed matrix, but it can also be used for constructing the sparse matrix in the transform space out of a dense matrix in the original domain by thresholding according to transform magnitudes. This would keep the size of the problem the same, but will make it sparse as was done previously by Beylkin [4] with the wavelet transform. In particular, diagonally dominant problems will not allow a simple reduction in size, but may allow the approximation of a full sized dense problem by a banded system in the Fourier domain.

Our general knowledge of the Fourier transform allows us to estimate feasibility of this approach by observing the overall distribution of values in a matrix. For example, it would be expected that sparse matrices will contain broad-banded frequency information, meaning that successful compression in frequency domain may not be possible. In crystallography, the Fourier space is sometimes called reciprocal space, meaning that the broad peak in real space will narrow in the Fourier space. There is a certain amount of flexibility concerning ordering of rows and columns in a matrix, as dictated by linear algebra. By reordering the rows and columns or doing any other allowed linear algebra manipulations prior to taking the transform, the frequency distribution in the Fourier transform of a matrix may be changed to suit our needs. This is not possible in regular signal processing, where the signal is defined as a time or space series.

A particular advantage of the present method is that computing solutions in transform space does not require construction of new linear algebra routines, but can use existing software for solving the problem in the transform domain.

Further work is needed in determining the objective criteria for matrix reduction in frequency space.

References